Dynamics of wave-pulse penetration into an evanescent region

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The initial problem of plasma wave dynamics in the presence of a sharp density jump that divides the space into transparent and opaque regions is studied. A wave packet is assumed to be initially localized in the transparent region. The transient process of field penetration beyond the density barrier during the wave packet reflection from the density jump is investigated. Signal velocity beyond the barrier is defined as the speed at which some small, but finite, value of the field amplitude appears in the evanescent region. This velocity, which is proved to be always less than the speed of light, is determined analytically for the case of quasi-Gaussian wave packet. Further insight into the field dynamics in opaque region is gained by considering a steplike initial wave packet.

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I. INTRODUCTION

As is known (see, for example, Ref. [1]), wave propagation in the approximation of geometrical optics (GO) is described by a set of equations similar to Hamilton's equations in mechanics,

$$\frac{d\vec{r}}{dt}=\frac{\partial\omega(\vec{k},\vec{r})}{\partial\vec{k}}\equiv\vec{v}_g,$$
$$\frac{d\vec{k}}{dt}=-\frac{\partial\omega(\vec{k},\vec{r})}{\partial\vec{r}}, \tag{1}$$

where the wave vector $\vec{k}$ plays the role of particle momentum, and the wave frequency $\omega(\vec{k},\vec{r})$ expressed as a function of $\vec{k}$ and $\vec{r}$ from a local dispersion equation plays the role of Hamiltonian function. The group velocity $\vec{v}_g$ which appears in Eqs. (1) is one of the fundamental quantities in the theory of wave propagation in dispersive media. We recall here some important aspects of the notion of group velocity following Ref. [2].

In the framework of Eqs. (1), the group velocity $\vec{v}_g$ represents the velocity along a ray trajectory on which the change of the wave vector $\vec{k}$ is governed by the second equation in Eqs. (1). This property of being the velocity of propagation of wave number perturbations constitutes one role of the group velocity. The other one consists in that the wave amplitude perturbations also propagate with the group velocity. In particular, if the initial wave packet is localized in space around $\vec{r}_0$, and its spectral amplitude is substantial only for $\vec{k}$ close to some value $\vec{k}_0$, then the resulting wave field is concentrated around the ray trajectory determined by Eqs. (1) with initial conditions $\vec{r}_0,\vec{k}_0$, the wave packet as the whole moving with the corresponding group velocity.

We should stress that the group velocity usage as in the last case suffers intrinsic inconsistency. Indeed, in order to speak about the coordinate of the wave packet, it should be localized in space at each moment of time. However, the group velocity is associated with a particular wave normal vector and, thus, assumes the wave packet to be narrow in the phase space. These two requirements are, strictly speaking, incompatible, since a function with a limited spectrum cannot be localized. Therefore, when dealing with spatially confined wave packets, one should keep in mind that their spectra contain arbitrary high harmonics that ensure zero field outside the localization region. By the way, this consideration already shows that it is meaningless to simultaneously think about band-limited filters and arbitrarily small wave amplitudes.

The true meaning of the GO concept of narrow (in the phase space) wave packets consists in that they have sharp spectral maxima and, consequently, spatial widths much larger than the wavelength. Thus, a consistent GO consideration of spatially localized wave packets requires the following scaling:

$$\lambda \ll L \ll \mathcal{L}, \tag{2}$$

where λ, L, and $\mathcal{L}$ are the characteristic values of the wavelength, the wave packet width, and the spatial scale of the problem, respectively. The inequalities (2) constitute the well known conditions of applicability for GO [3].

The concept of group velocity is usually used with reference to transparent media, i.e., in the case when both the frequency and wave vector connected by a dispersion relation are real quantities. (An investigation of the group velocity in an absorbing medium for the example of Gaussian wave packet propagation has been presented in Ref. [4], where the corresponding references may also be found.) According to GO, waves do not propagate in an area where the local dispersion relation gives an imaginary wave vector component for a fixed value of frequency. In this case, Eqs. (1) describe the reflection of a wave from an evanescent region. The basis for treating the wave frequency as being fixed is that, in a stationary medium, the wave frequency does not change. This argument is certainly valid if, from the

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very beginning, we look for a monochromatic and, thus, stationary solution to a problem. However, it is necessary to bear in mind that any initial problem is not stationary [5], and the physical requirements under which such a problem with localized initial conditions is near monochromatic are not trivial.

It is generally admitted that the description of electromagnetic field in the evanescent region in a way similar to that used for transparent region is problematic; in particular, the concept of signal velocity in opaque region is still under discussion [6,7]. Clearly, for a description of the field behavior in the evanescent region, it is necessary to use wave equations rather than the equations of GO. However, while using the wave equations, the basic characteristics of the field beyond a density barrier were obtained for a monochromatic and, thus, stationary problem. Obviously, such a consideration does not contain the dynamics of the process, as in a stationary problem, the dynamics are missing. Similar situation, as has been pointed out in Ref. [8], is in the problem of quantum tunneling: fixed energy tunneling described by stationary Schrödinger equation is well understood, while there are still open questions in nonstationary problem.

The present work is devoted to investigation of the field dynamics in the evanescent region, in particular, to determining the velocity v_s at which a field perturbation penetrates beyond the density barrier. We show that a physically meaningful definition of this quantity, which at the same time constitutes a means of measuring it, leads to finite values of v_s which are always less than the speed of light. We should note a closely related, both from physical and mathematical points of view, problem of particle tunneling time in quantum mechanics (see comprehensive discussions of the problem in Refs. [9–11] and references therein). Although the problem of quantum tunneling is out of the scope of the present work, some comments and comparisons related to our results are given where appropriate.

II. MODEL AND BASIC EQUATIONS

We consider a one-dimensional initial problem for transverse electromagnetic waves in a plasma with a density jump at the $x=0$ plane. In dimensionless variables in which the speed of light and the plasma frequency at $x<0$ are unity, this problem is described by the following equation for any component of the electromagnetic field:

$$\frac{\partial^2 f(x,t)}{\partial t^2} - \frac{\partial^2 f(x,t)}{\partial x^2} + \omega_p^2(x) f(x,t) = 0, \tag{3}$$

where

$$\omega_p^2(x) = \begin{cases} 1, & x<0 \\ \omega_+^2, & x>0, \end{cases}$$

and it is assumed that $\omega_+^2 > 1$. Introducing initial values of the field $f(x,t)$ and its derivative over time according to

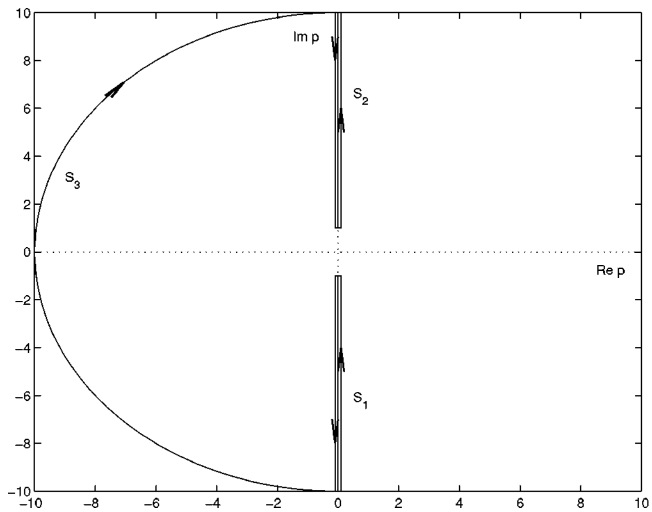

FIG. 1. Complex p plane.

$$\psi(x) = f(x,t)\big|_{t=0}, \quad \varphi(x) = \left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0},$$

and applying Laplace transformation over time to Eq. (3) we obtain

$$\frac{\partial^2 F(x,p)}{\partial x^2} - [\omega_p^2(x) + p^2] F(x,p) = -p\psi(x) - \varphi(x). \tag{4}$$

Here $F(x,p)$ is Laplace transform over time of the field $f(x,t)$ determined in the usual way (e.g., Ref. [12]). Note that the quantity in square brackets in Eq. (4) is equal, respectively, to

$$q^2 \equiv 1 + p^2, \quad x<0,$$
$$\kappa^2 \equiv \omega_+^2 + p^2, \quad x>0. \tag{5}$$

We now define q and κ as single-valued functions of the complex variable p,

$$q = \sqrt{1+p^2}, \quad \kappa = \sqrt{\omega_+^2 + p^2}, \tag{6}$$

where q and κ are the values of the square root with a positive real part. Obviously, the functions q and κ determined in this way are analytical functions of p in the right-half plane, i.e., at $\mathrm{Re}(p)>0$. However, if we introduce two branch cuts from $-i\infty$ up to $-i$ and from i up to $i\infty$ (see Fig. 1), then, given $\omega_+^2>1$, the functions determined above will be analytical functions of p in any area of the complex plane p which does not contain the branch cuts. On different banks of the cuts, the imaginary parts of the functions q and κ differ by signs, while their real parts are positive in the whole domain of analyticity. We also specify the asymptotic properties of the functions q and κ at $|p| \to \infty$,

$$q \simeq \kappa \simeq \begin{cases} p, & \mathrm{Re}(p)>0 \\ -p, & \mathrm{Re}(p)<0. \end{cases} \tag{7}$$

At the same time, on the part of imaginary axis p connecting the branch cuts, the functions q and κ are positive real quantities.

Having determined the functions q and κ, let us turn to fundamental set of the homogeneous equation (4). This set, which represents two linearly independent solutions of the homogeneous equation continuous at the point $x=0$, together with their first derivatives over x can be chosen in the form

$$H_1(x)=\begin{cases} e^{qx}, & x<0 \\ \dfrac{\kappa+q}{2\kappa}e^{\kappa x}+\dfrac{\kappa-q}{2\kappa}e^{-\kappa x}, & x>0, \end{cases}$$
$$H_2(x)=\begin{cases} e^{-qx}, & x<0 \\ \dfrac{\kappa-q}{2\kappa}e^{\kappa x}+\dfrac{\kappa+q}{2\kappa}e^{-\kappa x}, & x>0. \end{cases} \tag{8}$$

The corresponding Wronskian is equal to

$$W\equiv H_1(x)H_2'(x)-H_1'(x)H_2(x)=-2q, \tag{9}$$

where the prime means derivative with respect to x. Knowing the fundamental set of homogeneous equation it is possible to write the solution of the inhomogeneous equation (4) according to a general formula. The solution, finite at both $x\to-\infty$ and $x\to+\infty$, can be written as

$$F(x,p)=H_2(x)\int_{-\infty}^{x}\frac{H_1(x')r(x')}{W(x')}dx' +H_1(x)\int_{x}^{\infty}\frac{H_2(x')r(x')}{W(x')}dx'-H_1(x)\frac{\kappa-q}{\kappa+q} \times\int_{-\infty}^{\infty}\frac{H_1(x')r(x')}{W(x')}dx', \tag{10}$$

where $r(x)$ is the right hand side of Eq. (4),

$$r(x)\equiv-p\psi(x)-\varphi(x),$$

W is determined in Eq. (9), and all quantities r, H_1, H_2, and W depend on the parameter p of the Laplace transformation. The field $f(x,t)$ is expressed through $F(x,p)$ with the help of the inverse Laplace transformation (e.g., Ref. [12]),

$$f(x,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(x,p)e^{pt}dp \quad (\sigma>0). \tag{11}$$

Let us now simplify the solution (10) for $x>0$. Using the corresponding expressions for $H_1(x)$, $H_2(x)$, $r(x)$, and $W(x)$, and assuming that the initial perturbation is localized in the region $x<0$, we get

$$F(x,p)=\frac{pe^{-\kappa x}}{\kappa+q}\int_{-\infty}^{0}e^{qx'}\left[\psi(x')+\frac{\varphi(x')}{p}\right]dx', \tag{12}$$

where q and κ are the analytical functions of p determined in Eqs. (6). We assume that, at $t\to0$, $x<0$, the field $f(x,t)$ constitutes a wave packet composed of waves in which the frequency and wave vector are connected by the dispersion relation relevant for the propagation region $(x<0)$,

$$f(x,t\to0)=\int_{-\infty}^{\infty}\tilde{\psi}_1(k)e^{ikx-i\omega(k)t}\frac{dk}{2\pi} +\int_{-\infty}^{\infty}\tilde{\psi}_2(k)e^{ikx+i\omega(k)t}\frac{dk}{2\pi}, \tag{13}$$

where

$$\omega(k)=\sqrt{1+k^2}\,\mathrm{sgn}(k). \tag{14}$$

The first integral in Eq. (13) represents a wave packet propagating in positive direction of the x axis while the second integral corresponds to waves propagating in the negative direction of the x axis. As we assume that the wave packet is initially localized in the region $x<0$, the particular form of the function $\tilde{\psi}_2(k)$ should not influence the field in the $x\geqslant0$ region of present interest. Therefore, we can write

$$\tilde{\psi}_2(k)=\tilde{\psi}_1(k)\equiv\frac{\tilde{\psi}(k)}{2}. \tag{15}$$

Then,

$$\psi(x)\equiv f(x,t)|_{t=0}=\int_{-\infty}^{\infty}\tilde{\psi}(k)e^{ikx}\frac{dk}{2\pi},$$
$$\varphi(x)\equiv\left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0}=0. \tag{16}$$

Thus, with the assumptions (13) and (15), the second term in the square brackets in Eq. (12) vanishes. Let us designate by $\Psi(p)$ the integral corresponding to the first term in Eq. (12),

$$\Psi(p)=\int_{-\infty}^{0}e^{qx'}\psi(x')dx', \tag{17}$$

where $q=q(p)$ is the function defined above, so that $\Psi=\Psi[q(p)]$ is a composite function of p. Integral (17) is an analytical function of q for all $\mathrm{Re}(q)>0$. Since in the whole complex p plane outside the branch cuts the quantity q is an analytical function of p, and has $\mathrm{Re}(q)>0$, in the same area of p plane the function $\Psi(p)$ appears to be an analytical function of p. In terms of the notation introduced, the expression for the field which follows from Eqs. (11) and (12) can be rewritten in the form

$$f(x,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{pe^{-\kappa x}}{\kappa+q}\Psi(p)e^{pt}dp. \tag{18}$$

We should notice that the quantity $(\kappa+q)$ is not equal to zero anywhere on the complex plane p, which is easily verified with the help of definitions (6).

III. FIELD PENETRATION INTO AN EVANESCENT REGION IN THE CASE OF A QUASI-GAUSSIAN INITIAL WAVE PACKET

A. Expression for the field

Let the field at $t=0$ be expressed as

$$\psi(x)\equiv f(x,t=0)=A(x-x_m)e^{ik_0x}, \tag{19}$$

where $A(x-x_m)$ is a slowly varying amplitude localized in the region $x<0$ around x_m and having a characteristic scale L obeying the requirements

$$2\pi/k_0\ll L\ll|x_m|. \tag{20}$$

We will assume that, in the region where $A(x-x_m)$ is essentially distinct from zero, it has the form

$$A(x-x_m)\simeq e^{-(x-x_m)^2/L^2}. \tag{21}$$

We cannot, however, consider this presentation to be valid for all x. This would lead to incorrect properties of the function $\Psi(p)$ (17) at $|p|\to\infty$ because the expression (21), although exponentially small, is not zero at $x>0$. Nevertheless, for the sake of shortness, we will omit the prefix ''quasi'' later on.

According to the problem under discussion, we will assume that the wave frequency ω_0 corresponding to the wave vector k_0 in the region $x<0$ is lower than the plasma frequency at $x>0$, i.e.,

$$\omega_0\equiv\sqrt{1+k_0^2}<\omega_+. \tag{22}$$

The function $\Psi(p)$ (17), which defines the field at $x>0$ according to the formula (18), is thus equal to

$$\Psi(p)=\int_{-\infty}^{0}e^{qx'+ik_0x'}A(x'-x_m)dx'. \tag{23}$$

Proceeding to the analysis of the field in the evanescent region determined by Eqs. (18) and (23) let us assume that $A(x'-x_m)\equiv0$ for $x>x_0$, i.e., $x_0<0$ is the coordinate of the leading edge of initial wave packet. Then, the quantity (23) permits the following estimation:

$$|\Psi(p)|<e^{\mathrm{Re}(q)x_0}\int_{-\infty}^{0}|A(y+x_0-x_m)|dy$$
$$<Me^{\mathrm{Re}(q)x_0},\quad M=\mathrm{const}<\infty, \tag{24}$$

where we have assumed that the amplitude of the initial wave packet is an absolutely integrable function. Using Eq. (24) and relations (7), we find the asymptotic behavior of the integrand in Eq. (18) at $|p|\to\infty$ and $\mathrm{Re}(p)>0$, i.e., in the right-half plane,

$$\left|\frac{pe^{-\kappa x+pt}}{\kappa+q}\Psi(p)\right|<\frac{M}{2}e^{\mathrm{Re}(p)(t-x+x_0)}.$$

We see that for

$$t<x-x_0\equiv x+|x_0|, \tag{25}$$

the integrand decreases exponentially at $|p|\to\infty$ and $\mathrm{Re}(p)>0$. Thus the contour of integration can be closed in the right-half plane. Since the Laplace transform is an analytical function of p in the right-half plane, the total integral is identically equal to zero. The quantity $x+|x_0|$ represents the distance from the packet leading edge at $t=0$ to the point x in the evanescent region. Since x_0 and x are arbitrary (provided that $x_0<0$ and $x>0$), and in dimensionless units the speed of light is equal to unity, the result above shows that, either in the transparent or in the evanescent region, the signal does not propagate with a velocity exceeding the speed of light. This general result, which arises naturally in our consideration, is the direct consequence of asymptotic relations (7) connected with Eq. (3). We would not find this effect if, instead of using Lorentz-invariant equation (3), we used Schrödinger equation, which, in a sense, is an expansion of Eq. (3) valid only for small $(k-k_0)^2$. Thus, the result above is a relativistic effect. We should mention that superluminal or infinite velocities could be ''found'' in many cases other than Schrödinger equation, when nonrelativistic relations are used out of the frame of their validity. For example, Newton's law leads to unlimited increase of a charged particle velocity in a constant electric field; less trivial example is that according to classical thermal conductivity equation, a temperature perturbation propagates with infinite velocity, etc.

Let us now turn to the general case $t>0$, when t is not restricted by inequality (25). As the integration in Eq. (23) is performed only over the region $x<0$, the quantity $\Psi(p)$ drops exponentially at $|p|\to\infty$ and $\mathrm{Re}(p)<0$, since the corresponding values of q obey the requirements $|q|\to\infty$ and $\mathrm{Re}(q)>0$ [see Eq. (7)]. Thus, at $t>0$, the integrand in Eq. (18) tends to zero exponentially at $|p|\to\infty$ and $\mathrm{Re}(p)<0$. Nevertheless, we cannot close the contour of integration in the left-half plane due to the presence of branch cuts (see Fig. 1). It is possible, however, to deform the contour of integration so that it does not intersect the branch cuts anywhere and to perform the integration over the contour $S_1+S_3+S_2$ shown in Fig. 1. As we have seen above, the integral over the contour S_3 tends to zero, so the evaluation is reduced to integration over the contours S_1 and S_2, i.e., over the banks of cuts. It is convenient to transfer from integration over p to integration over the variable q. In doing this, one should remember the following: the functions q and κ are single-valued analytical functions of p outside the branch cuts. However, there is no one-to-one relation between these functions, as the values p and $-p$ correspond to the same values of q and κ. One should bear this in mind when expressing the functions p and κ through q in different parts of the complex plane of the variable p. We will write the complex quantities p, q, and κ as $p=p_1+ip_2$, $q=q_1+iq_2$, $\kappa=\kappa_1+i\kappa_2$. Obviously, on the banks of cuts, the values of p_1 and $q_1>0$ are infinitesimal, i.e., p and q are close to purely imaginary numbers. The quantity κ also has an infinitesimal positive real part on the banks of cuts at $|p_2|>\omega_+$, while at $1<|p_2|<\omega_+$ it is close to real positive quantity. These properties, which follow from definitions (5),(6) allow us to express p and κ as functions of q on the contours S_1 and S_2. Thus, for the quantity p we have

$$p=\begin{cases} -i\sqrt{1+q_2^2} & \text{for } \operatorname{Im}(p)<0 \text{ (contour } S_1), \\ i\sqrt{1+q_2^2} & \text{for } \operatorname{Im}(p)>0 \text{ (contour } S_2). \end{cases}$$

As for the quantity κ, according to Eqs. (5) and (6), $\kappa=\sqrt{(\omega_+^2-1)+q^2}$. Here, however, we cannot take $q_1=0$ on the contours of integration, since the requirement $\operatorname{Re}(\kappa)>0$ does not permit us to determine the branch of the square root corresponding to those q values for which the quantity κ is purely imaginary. Therefore, for an exact definition of the square root, we should take into account that the quantity q has, though infinitesimal, positive real part. This leads to a definition of κ valid on both contours of integration

$$\kappa=\sqrt{(\omega_+^2-1)-q_2^2+i\varepsilon\,\operatorname{sgn}(q_2)},\quad \operatorname{Re}(\kappa)>0. \tag{26}$$

Small additive $i\varepsilon\operatorname{sgn}(q_2)$ does not play a role for $q_2^2<(\omega_+^2-1)$ when κ is a positive quantity but allows us to correctly determine the sign of the imaginary part of κ for $q_2^2>(\omega_+^2-1)$.

As we have seen, the integration in Eq. (18) can be reduced to integrals over the banks of cuts on the plane p (Fig. 1), where the quantity q is purely imaginary: $q=iq_2$. When evaluating the quantity $\Psi(p)$ (23) for imaginary values of q, we can already extend the integration to the whole x axis and use the expression (21) everywhere, since under conditions (20) the region $x>0$ brings an exponentially small contribution to the integral. We then get

$$\Psi(p)=\sqrt{\pi}Le^{-(q_2+k_0)^2L^2/4+i(q_2+k_0)x_m}. \tag{27}$$

Using the expression (27) and transfering to new variable of integration $k=-q_2$ we obtain from Eq. (18),

$$f(x,t)=\frac{L}{2\sqrt{\pi}}\int_{-\infty}^{\infty}dk\frac{k}{[k+i\kappa(k)]}e^{-(k-k_0)^2L^2/4}\times e^{-\kappa(k)x-i(k-k_0)x_m-i\omega(k)t}+\frac{L}{2\sqrt{\pi}}\int_{-\infty}^{\infty}dk\frac{k}{[k+i\kappa(k)]}\times e^{-(k-k_0)^2L^2/4}e^{-\kappa(k)x-i(k-k_0)x_m+i\omega(k)t}, \tag{28}$$

where the first and second integrals correspond, respectively, to integration over the contours S_1,S_2; $\omega(k)$ is given by Eq. (14), and $\kappa(k)=\kappa(-q_2)$, where $\kappa(q_2)$ is determined in Eq. (26).

Proceeding to the analysis of the field (28), we notice that the characteristic scale of variation of the factor $\exp[-(k-k_0)^2L^2/4]$ is of the order of $\sqrt{\pi}/L$, while the fast varying exponent $\exp[-i(k-k_0)x_m]$ oscillates with a much smaller period $2\pi/|x_m|$ [see Eq. (20)]. As the variation of the total phase in the second integral in Eq. (28) for $k\sim k_0$ is fast for all $t>0$, its contribution to the field is exponentially small, so that the field at $x>0$ is approximately determined by

$$f(x,t)\simeq\frac{L}{2\sqrt{\pi}}\int_{-\infty}^{\infty}dk\frac{k}{[k+i\kappa(k)]}e^{-(k-k_0)^2L^2/4}\times e^{-\kappa(k)x-i(k-k_0)x_m-i\omega(k)t}. \tag{29}$$

The integral in Eq. (29) can be evaluated with the help of saddle-point technique using the fact that k_0x_m, k_0v_gt, and k_0L are large parameters. The saddle point is specified by the equation

$$(k_s-k_0)L^2+2i[x_m+v_g(k_s)t]=0, \tag{30}$$

which determines the point k_s as the function of t and the parameters k_0, L, and x_m. The wave group velocity v_g which enters Eq. (30) is equal to

$$v_g\equiv\frac{\partial\omega}{\partial k}=\frac{k}{\sqrt{1+k^2}}.$$

The integral (29) contains the fast oscillating factor and, thus, is exponentially small unless $|x_m+v_g(k)t|\lesssim L/\sqrt{\pi}\ll|x_m|$. (We remind the reader that $x_m<0$.) The analysis of Eq. (30) shows that the position of the saddle point on the complex k plane is controlled by the parameter that we denote by α and which is of fundamental importance in our problem,

$$\alpha=\frac{|x_m|}{k_0L^2}. \tag{31}$$

The physical meaning of the parameter α can be understood from the well known GO equation for the complex wave amplitude $\tilde{A}$ including dispersive terms (see, for example, Ref. [13]),

$$i\left(\frac{\partial\tilde{A}}{\partial t}+v_g\frac{\partial\tilde{A}}{\partial x}\right)+\frac{1}{2}\frac{\partial v_g}{\partial k}\frac{\partial^2\tilde{A}}{\partial x^2}=0,$$

which shows that the dispersive spreading of the wave packet becomes important for $t_d\sim k_0L^2/v_g$, while the time of wave packet propagation up to the density jump is $t_p\sim|x_m|/v_g$. We thus see that the parameter α controls the dispersive spread of the wave packet before it reaches the point $x=0$. In the following, we will assume that

$$\alpha\ll 1, \tag{32}$$

i.e., the dispersive spread is not important. In this case one gets from Eq. (30),

$$(k_s-k_0)=-2i\frac{x_m+v_{g0}t}{L^2+2iv'_{g0}t}, \tag{33}$$

where

$$v_{g0}=v_g(k_0),\quad v'_{g0}=\left.\frac{\partial v_g}{\partial k}\right|_{k=k_0}.$$

We see that the saddle point is close to the real axis, the deviation of the real part of k_s from k_0 being of the second order in parameter α. The expansion of the fast varying part of the phase in Eq. (29) around k_s has the form

$$-\frac{(k-k_0)^2L^2}{4}-i(k-k_0)x_m-i\omega(k)t$$

$$=-i\omega_0 t-\frac{(x_m+v_{g0}t)^2}{L^2+2iv'_{g0}t}-\left(\frac{L^2}{4}+\frac{iv'_{g0}t}{2}\right)(k-k_s)^2, \quad (34)$$

where $\omega_0=\omega(k_0)$. Substituting $k=k_s$ into all slowly varying quantities in Eq. (29) and performing the integration we obtain in a straightforward way

$$f(x,t)\simeq\frac{L}{\sqrt{L^2+2iv'_{g0}t}}\frac{k_s}{[k_s+i\kappa(k_s)]}\times\exp\left[-\frac{(x_m+v_{g0}t)^2}{L^2+2iv'_{g0}t}\right]e^{-\kappa(k_s)x-i\omega_0 t}, \quad (35)$$

where the square root of the complex quantity has a positive real part. As follows from Eq. (35), the field is essentially distinct from zero only for $t\sim -x_m/v_{g0}$ when $k_s\simeq k_0$ [see Eq. (33)]. In the lowest order in parameter α, the expression for the field amplitude takes the form

$$|f(x,t)|\simeq\frac{k_0}{\sqrt{\omega_+^2-1}}e^{-\kappa(k_0)x-(x_m+v_{g0}t)^2/L^2}. \quad (36)$$

B. Signal velocity in the evanescent region

Expression (36) permits us to determine the velocity of signal penetration beyond the barrier. We define the time of signal arrival at a point x as the time when the signal amplitude reaches a certain measurable level $A_{\min}$. When it happens, the corresponding value of the exponent in Eq. (36), which we denote by ϵ, is equal to

$$e^{-\kappa(k_0)x-(x_m+v_{g0}t)^2/L^2}\equiv\epsilon=\frac{A_{\min}\sqrt{\omega_+^2-1}}{k_0}. \quad (37)$$

We should mention that, by definition, $\epsilon<1$. Equation (37) determines the signal arrival time at the point x while the derivative of the inverse function gives the velocity of the signal in the region $x>0$,

$$\frac{dx}{dt}\equiv v_s=\frac{2v_{g0}}{\kappa_0 L}\sqrt{\ln\left(\frac{1}{\epsilon}\right)-\kappa_0 x}. \quad (38)$$

We see that the signal velocity beyond the density barrier is essentially different from that in the transparent region. First, it decreases with x up to zero, in accordance with the fact that, at large x, the field amplitude never reaches the chosen level. Second, the signal velocity depends, although logarithmically, on the chosen level $A_{\min}$, so that lower values of signal amplitude propagate with higher velocities. It is important to note that the value of ϵ cannot be chosen arbitrary small. Indeed, according to Eqs. (19) and (21) the field at $x=0$, $t=0$ is equal to

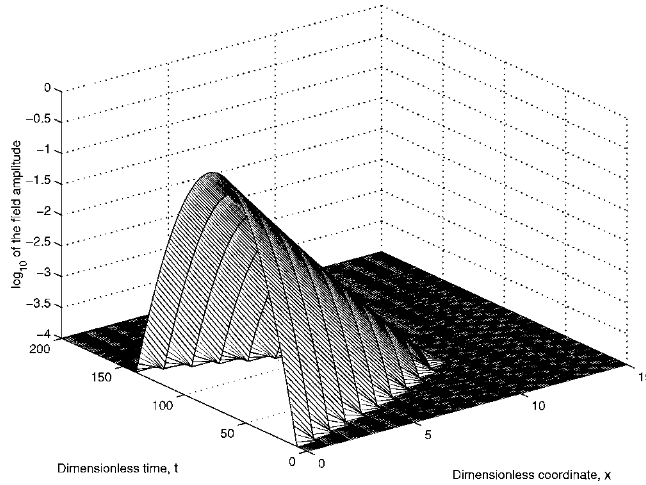

FIG. 2. Surface of the field amplitude above the (x,t) plane for the Gaussian initial wave packet.

$$f(x=0,\ t=0)\simeq\exp(-x_m^2/L^2). \quad (39)$$

As we assume that at $t=0$ the field is localized in the region $x<0$, the minimum level of measurable signal should be much larger than the value (39), i.e.,

$$\epsilon\gg\exp(-x_m^2/L^2),$$

so that

$$v_s<\frac{2v_{g0}|x_m|}{\kappa_0 L^2}\ll\frac{2v_{g0}k_0}{\kappa_0},$$

where the relations (31) and (32) have been taken into account. Thus, the velocity of signal is always less than the speed of light.

Recently, there has been a considerable discussion of the signals related to the evanescent mode (see Ref. [14] and other papers from Ref. [11]). The central point in this discussion is whether or not those signals can travel with the velocity exceeding the speed of light. The paper [14] claims such a possibility, which has supposedly been demonstrated. At the same time, we have shown that the signal velocity introduced above is always less that the speed of light. To clear up this point, let us consider the dynamics of the field penetration into the evanescent region in more detail. Figures 2–4 show the surface of the field amplitude in the evanescent region, as well as the cross sections of this surface representing the field amplitude as the function of time for various coordinates, and the field amplitude as the function of coordinates for various moments of time. Figures 2–4 correspond to the following values of parameters:

$$x_m=-50,\ \ L=15,\ \ k_0=1,\ \ \omega_+^2=3.$$

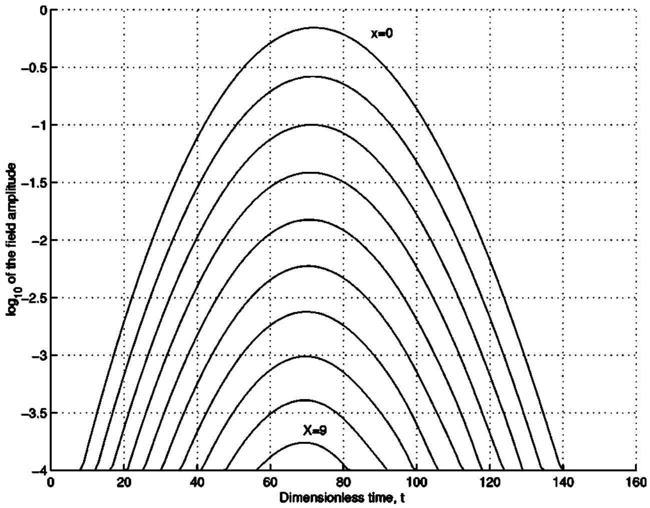

FIG. 3. Field amplitude as the function of time for various coordinates $x=1,2, \ldots ,9$, for the Gaussian initial wave packet.

We see that the field dynamics (which, of course, exists in nonstationary problem, as the field amplitude at each point depends on time) has nothing to do with propagation. We suspect that the conclusion about the instantaneous traverse of the evanescent region by the signal is connected with a misinterpretation of some plot similar to that shown in Fig. 3, and with the extension of an intuitive concept of the mechanical velocity to nonlocal process. In general, to determine a velocity, we need to know two spatial positions of the object at two moments of time. A mass point represents the simplest example which casts no doubt about the mass point position at each moment of time. A wave packet with a characteristic width less than the distance between two spatial points can be treated in a similar way. But in the case of evanescent region, we deal with the opposite situation. Here the characteristic spatial scale is less than the wave packet width because the wave vanishes on the length of the order of several wavelengths, while the initial wave packet width is much larger than that [see Eq. (20)]. Thus, Fig. 3 by no means implies that the signal reaches all spatial points in the evanescent region at the same time. This becomes clearer if we consider Fig. 4: looking, for example, at the boldface line corresponding to dimensionless time $t=70$, it is impossible to say at which point the signal is at this moment, as the signal is spread over the whole evanescent region. The instantaneous traverse of the evanescent region, and the corresponding infinite velocity pointed out in Ref. [14], would require the absence of the field in the evanescent region at the preceding moment, which, of course, is not the case. Moreover, if one defines the signal position at some moment t as the point corresponding to the maximum spatial amplitude, then the signal would always be at the point $x=0$ and thus not propagating at all.

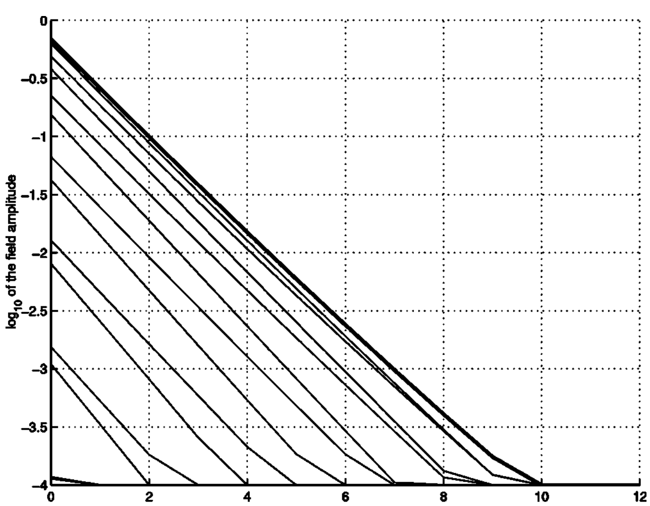

FIG. 4. Field amplitude as the function of x for various times $t=10,20, \ldots ,140$, for the Gaussian initial wave packet.

IV. PECULIARITIES OF THE FIELD DYNAMICS IN THE MODEL OF A STEP-LIKE INITIAL WAVE PACKET

As the question about signal velocity in the evanescent region is of fundamental importance, to get more insight into this problem we will consider another model of the initial wave packet, namely,

$$f(x,t=0)\equiv\psi(x)=R(x)e^{ik_0x}, \tag{40}$$

where the function $R(x)$ is equal to

$$R(x)=\begin{cases}1 & \text{for } (x_m-l)<x<(x_m+l),\\ 0 & \text{for } x<(x_m-l)|x>(x_m+l).\end{cases} \tag{41}$$

Function $R(x)$ (41), which determines the initial wave packet envelope, represents a step of a finite width in the x space. As in the case of Gaussian wave packet, the quantity $x_m<0$ is equal to the coordinate of the wave packet center, $l>0$ is its half-width, the coordinate of the packet front (x_m+l) lies in the region $x<0$, and the relation (22) is also assumed. The expression (18) for a field in the evanescent region remains valid, while the function $\Psi(p)$ now takes the form

$$\Psi(p)=2\frac{\sinh[(q+ik_0)l]}{(q+ik_0)}e^{(q+ik_0)x_m}, \tag{42}$$

where sinh stands for hyperbolic sine.

The field determined by Eqs. (42) and (18) is equal to

$$f(x,t)=\frac{e^{ik_0x_m}}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\sinh[(q+ik_0)l]}{(q+\kappa)(q+ik_0)}e^{qx_m-\kappa x+pt}pdp. \tag{43}$$

Since the integrand in Eq. (43) is an analytical function of p in the right-half plane, and at $|p|\to\infty$ is asymptotically equal to $e^{p(t+x_m+l-x)}/p$ [see Eq. (7)], we immediately find that at $t<x+|x_m+l|$ the integral is equal to zero. Thus, in this model we also meet with the fact that, in both the regions $x<0$ and $x>0$, the signal moves with a velocity not larger than the speed of light.

For $t>x+|x_m+l|$, in evaluation of the field (43) we may follow the same steps as in the case of Gaussian wave packet, thus arriving at

$$f(x,t)\simeq\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin[(k-k_0)l]}{(k-k_0)[k+i\kappa(k)]}e^{-i(k-k_0)x_m-\kappa(k)x-i\sqrt{1+k^2}t}k\,dk, \tag{44}$$

where $\kappa(k)=\sqrt{(\omega_+^2-1)-k^2-i\varepsilon\,\mathrm{sgn}(k)}$, $\mathrm{Re}[\kappa(k)]>0$. Due to Eq. (22), κ is a real positive quantity for $k=k_0$, which is a manifestation of the fact that the region $x>0$ is opaque for the frequency $\omega_0=\sqrt{1+k_0^2}$.

The field $f(x,t)$ at $x>0$ is equal to zero not only for $t<x+|x_m+l|$, as we have seen above, but also for $t\to\infty$, since the integrand in Eq. (44) is smooth and finite on the whole k axis, tends to zero for $|k|\to\infty$, and contains an oscillating factor whose period tends to zero at $t\to\infty$. The fact that the field in the evanescent region vanishes at $t\to\infty$ is connected with that we consider the density barrier which is semi-infinite in space, i.e., that extends in the whole region $x>0$. This excludes an asymptotic wave tunneling, so the process that we study is purely transient. Apart from dealing with a type of telegraph equation rather than with Schrödinger equation, this is another point that differs our study from investigations of quantum tunneling through a finite width potential barrier [8,15].

Obviously, the intermediate values of time, i.e., $t\gtrsim x+|x_m+l|$ are of the main interest for investigating the transient process of field penetration beyond the density barrier. For the time interval which includes the above mentioned values, the integral (44) has been calculated numerically. The surface of the field amplitude $A(x,t)=|f(x,t)|$ is shown in Fig. 5 for the following values of the model parameters:

$$x_m=-20,\quad l=15,\quad k_0=1,\quad \omega_+^2=3.$$

At first glance the ‘‘rectangular’’ wave packet model (40), (41) seems quite suitable for defining the velocity of field penetration beyond a density barrier. Really, the spectrum of the initial perturbation has a pronounced maximum at $k=k_0$, which corresponds to $\omega_0<\omega_+$ in the region $x<0$; the field is initially localized and has a sharp leading edge at $x=x_m+l<0$. However, if we define the signal velocity in the evanescent region in the same way as for Gaussian wave packet, i.e., according to the time of finite signal amplitude arrival at some point, we will find that, in both the transparent and evanescent regions, the signal propagates with the velocity of light (see Fig. 5). Just the fact that the signal reaches the point $x=0$ at $t=|x_m+l|$, thus propagating in the transparent region with the speed of light rather than with a group velocity $v_{g0}<1$, gives the key to understanding this result. The point is that the packet (40),(41) contains high harmonics with $k\gg k_0$ with significant amplitudes for which the density jump at the point $x=0$ does not represent a ‘‘potential barrier’’ and which propagate at all x with a velocity close to the speed of light.

As we see from Fig. 5, the field amplitude A at $x>0$ has a quite complicated structure. Figure 6 shows the amplitude A as the function of time for several coordinates in the evanescent region. Not surprisingly, for $x=0$ the plot is centered on $t=|x_m|/v_{g0}$. However, even at $x=0$ there is a significant spread of the wave packet as compared to the initial shape. Although a characteristic width of the plot $\Delta t\sim 2l/v_{g0}$ is well explained by propagation features in the region $x<0$, the essential influence of the $k\neq k_0$ harmonics is clearly seen. An important feature of the field dynamics in the region $x>0$ in the case of a steplike initial wave packet is that it evidently shows signs of propagation. This is related to the fact that the integral in Eq. (44) is slowly converging, which is a mathematical manifestation of the influence of the high harmonics mentioned above. The influence and eventual dominance of high harmonics in the framework of Schrödinger equation has been pointed out by Hartman [16], (see also Refs. [15,17]). Thus, the field at $x>0$ is determined by a competition between $k\simeq k_0$ harmonics which dominate the

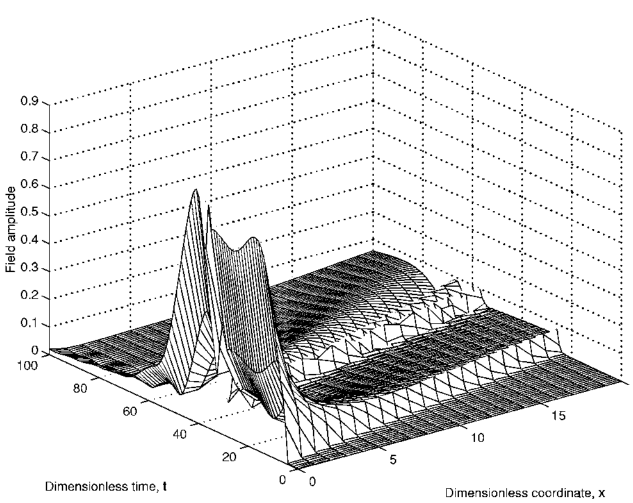

FIG. 5. Surface of the field amplitude above the (x,t) plane for the steplike initial wave packet.

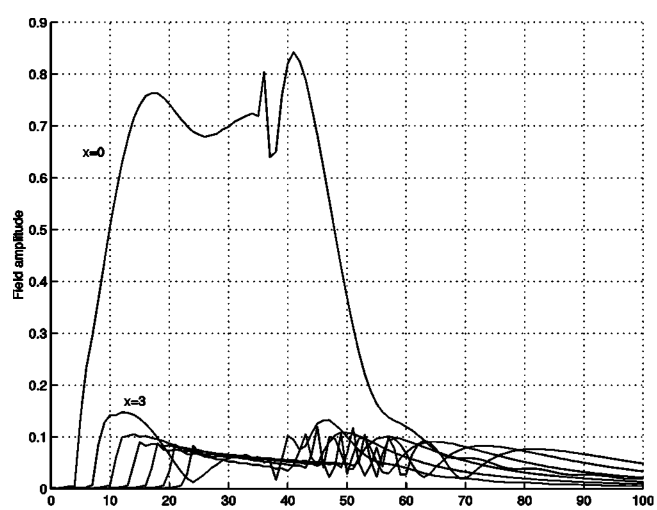

FIG. 6. Field amplitude as the function of time for various coordinates $x=0,3,\ldots,18$, for the steplike initial wave packet.

initial field but attenuate heavily in the evanescent region, and $|k-k_0| \gg 1/l$ harmonics whose amplitudes are smaller but which suffer less or no attenuation in the region $x>0$. This makes the problem extremely sensitive to the spectrum of initial perturbation, especially to the way how it drops at $k \to \infty$, which has been pointed out in Ref. [18]. At $x=0$, the field is mainly determined by the $k \simeq k_0$ harmonics which undergo no attenuation in the $x<0$ region. However, for $x>0$, the bulk of high harmonics, some of which can propagate in the region $x>0$, play the dominant part, causing the propagation features to appear. We should underline that, in the context of field penetration into the evanescent region, the term "attenuation" used above describes the field behavior which is not related to a true absorption of energy.

Main features of the time dependence of the local field amplitude in the evanescent region shown in Fig. 6, namely, the presence of two maxima, oscillations after the second maximum, and final vanishing at large time are quite similar to those found in Ref. [8] for the case of finite width potential barrier in the context of Schrödinger equation (see Fig. 7 from Ref. [8]). It is not surprising, of course, since the features of the field dynamics which are not determined by high harmonics should be alike for Schrödinger and telegraph equations.

V. SIGNAL VELOCITY FOR AN EXTENDED, FINITE BANDWIDTH SOURCE VERSUS SUPERLUMINAL PROPAGATION

As we have seen above, in the case of a Gaussian initial wave packet, the local amplitude maxima over time are reached at all points in the evanescent region simultaneously. In the case of a steplike initial wave packet, the amplitude maximum at $x=0$ is reached even later than at some points $x>0$. Phenomena of this type are sometimes referred to as "superluminal signal propagation" [14]. We proceed with a general discussion of this issue pertaining to both Gaussian and steplike initial wave packets, in the framework of classical electrodynamics, to which the contents of the present paper are related. First of all, we notice that no signal reaches any point in either transparent or evanescent region faster than the light would reach this point from the leading edge of the wave packet. Also, any given value of the field amplitude propagates in a "proper" way, namely, it first reaches the points that are closer to the wave packet front. However, there is a tendency to consider the arrival of the amplitude maximum as the arrival of some information. But even this, on closer examination, does not contradict causality or relativity principles. One should only realize that, in the problem under discussion, we deal with an extended initial source with a dimension exceeding that of the region where the signal is registered. Another essential point, which is particularly important for a steplike initial wave packet, is that a localized field always contains high harmonics. Although we consider an initial problem, we deal with the second order equation over time, and the initial conditions also include the time derivative of the field. Thus, in fact, we have high space/time harmonics in the initial signal. Furthermore, it is completely wrong, for example, for a Gaussian wave packet, to say that some signal (or information) associated with the amplitude maximum comes first to the point $x=0$ and then in no time reaches the whole evanescent region. Since these maxima decrease exponentially with the distance, they cannot be considered as the same signal. (The observed reshaping of a wave packet, or deformation of a pulse should not be mixed with propagation [15].) Thus, we can only say that some information sent from an extended source by means of a finite (or even infinite) band width wave packet reached the whole evanescent region simultaneously. Under the conditions that (1) the time at which the signal reaches any point is larger than the distance from the wave packet leading edge at $t=0$ divided by the speed of light, and (2) the spatial dimension of the evanescent region is smaller than that of the source, both of which are met, the simultaneous appearance of local field amplitude maxima at all points in the evanescent region does not contradict either relativity or causality principles. This conclusion is based on the detailed analysis of particular physical model. Strong arguments against the interpretation of experimental or theoretical results that assumes violation of Einstein causality have been given in Ref. [6] based on most general physical principles.

VI. CONCLUDING REMARKS AND SUMMARY OF RESULTS

Some of the recent works that investigate field evolution in an evanescent medium consider boundary problem for a point source within this medium [7,17]. Application of an effective filter to the source or receiver considered in Refs. [7,17] gives an important insight into the field nature in the evanescent medium. In particular, it permits to reduce the magnitude of forerunner as compared to monochromatic front, which helps detection of the latter (see the references above for details). Since in the present paper we study initial problem with an extended wave packet originally localized in transparent region, direct comparison of our results with those obtained in the above cited papers is hardly possible. In general, we believe that initial problem that we consider is more suitable and clear for correct definition of signal velocity in the evanescent region.

Inasmuch as in this study we are mainly interested in determining the velocity of the signal as the whole, and in finding conditions under which it is possible, we always deal with the total spectrum of the signal. Not surprisingly, the velocity found [see expression (38)] is not universal, but depends on specific initial conditions. More precisely, the signal velocity depends on v_{g0} (or k_0), which is the wave characteristic in transparent region, on κ_0, which accounts for the effective potential height, and also on L, which characterizes initial spectrum, or equivalently, initial shape of the wave packet envelope.

It is well understood and seems to be generally accepted that superluminal propagation of the front (or leading edge) of a signal in the framework of Schrödinger equation is connected with the fact that this equation, in contrast to Maxwell's equations, is not Lorentz invariant [15]. However, the question of superluminal velocity of a signal as a whole is still under discussion, and so is the definition of this velocity.

A number of suggestions have been listed in Ref. [6]. Obviously, in the framework of Schrödinger equation this question is meaningful only in the case when the field in the evanescent region is not dominated by high harmonics. As we have seen above, if the initial spectrum does not drop fast enough, e.g., in the case of steplike initial conditions, the signal velocity is difficult to determine in the context of telegraph equation though.

The idea of superluminal velocity arises from the association (in fact, unsubstantiated) of the time at which a local maximum of the field amplitude appears at some point x in the evanescent region with the time of signal arrival. Such an association typical of the wave propagation in transparent media assumes silently that the wave packet amplitude is transported in space. In the evanescent region, where the local amplitude maximum decreases exponentially in space, this conception has no physical grounds. Indeed, the wave packet position at a time t (or the most probable position of a particle) might be connected with a spatial maximum of the wave packet amplitude (or the wave function amplitude) at this time. However, it is wrong to associate the amplitude maximum over time observed at some point with the wave packet (or particle) position, as the wave packet amplitude may be larger (or even much larger) at some other point at the same time. Thus, signal velocity in the evanescent region needs to be rigorously defined, as has been underlined by many authors (e.g., Refs. [6,7]).

Towards this aim, and in order to investigate the dynamics of electromagnetic field penetration into the evanescent region beyond a density jump in a plasma, the initial wave problem has been solved in a frame of telegraph equation, with initial conditions in the form of a wave packet localized in front of the density barrier. Since the wave field in the evanescent region is initially absent, and since, during the process of reflection from the density jump, the field partly penetrates through, there should be a physically meaningful quantity which describes the velocity of penetration. To define a physical quantity means to determine the way how this quantity should be measured [19]. We suggest defining signal velocity as the speed at which some small but finite level of the field amplitude appears in the evanescent region. This definition permits the determination of the velocity of the field penetration into the evanescent region theoretically [see the exact analytical expression (38)] and implies a way for its unambiguous experimental measurements. We show that the velocity determined in such a way never exceeds the speed of light.

Delgado and Muga [10] have analyzed the possibility to measure anomalously short tunneling time in the case when initial wave packet is localized close (on the scale of the barrier width d) to a potential barrier, and arrived at a negative conclusion. They have also revealed essential and well motivated physical parameters of the problem. We should formulate clearly our conclusions on this matter for the problem studied in this paper. The first natural and common requirement is the initial wave packet localization in front of the barrier. For a steplike packet, this assumes that, in considered geometry, the packet leading edge is initially located to the left of the barrier. For a Gaussian initial wave packet, the localization in front of the barrier implies inequalities (20). Although we consider semi-infinite barrier ($d\rightarrow\infty$), inequalities (20) mean that, in a sense, the wave packet is initially located far from the density barrier (as compared to its own width L). For a steplike packet, which is only complimentary example in our study, we did not perform any comprehensive analysis of the field dependence on the model parameters, and did not find a case when the field in the evanescent region is not dominated by high harmonics. In the considered example of parameters (see comments to Fig. 5), the signal velocity in the evanescent region consistent with our definition is equal to the speed of light.

Considering Gaussian wave packet, we have established condition (32) for parameter α (31) that, together with Eqs. (20), permits to introduce the velocity for the signal as the whole relevant to the evanescent region. In physical terms, these conditions require that, on one hand, the wave packet maximum is situated far enough from the density jump in order for a signal to be initially localized in transparent region; but on the other hand, it is localized close enough to the density jump in order for dispersive spread before reaching the barrier to be negligible. These two conditions are compatible provided that $kL\gg 1$, i.e., the wavelength is much shorter than a characteristic wave packet width. When these conditions are met, the saddle point k_s which determines the field dynamics in the evanescent region is close to the central wave number k_0 of the initial wave packet. For Gaussian wave packet, not only the signal velocity is less than the speed of light, but it also has quite a specific character: it depends on the distance in the evanescent region, as well as on minimum measurable signal amplitude.

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